The formulas of the preceding section hence remain valid; it is only necessary to assume $K(-{}^3/_2) = 1$ therein. For example, evaluating the functions

$$D_2^{**}(-2) = 2, \quad K^+(-2) = \pi^{-1/2}, \qquad C = -T[4\sqrt{\pi}]^{-1}$$

according to (1.46), (2.1), (1.47), and substituting them into (1.49), we obtain the known formula for the indentation of a flat circular die into an elastic half-space

$$u_0 = T(1-\sigma)\ [4\pi\ G]^{-1}$$

The normal stress distribution under the die is also found easily from (1.44)

$$\sigma_\theta = -\frac{1}{2\pi i}\int_L \frac{Tr^{-\nu-3}\,d\nu}{4\sqrt{\pi}K^+(\nu)} = \frac{T}{8i\sqrt{\pi}}\int_L \frac{r^{-\nu-3}\,d\nu}{\cos({}^1/_2\pi\nu)\,\Gamma(2+{}^1/_2\nu)\,\Gamma({}^1/_2-{}^1/_2\nu)} =$$

$$= -\frac{T}{2\pi}\sum_{k=0}^{\infty}(ir)^k P_k(0) = -\frac{T}{2\pi\sqrt{1-r^2}}$$

In conclusion, the author is grateful to Ia. S. Ufliand for discussing the research, and for useful remarks.

BIBLIOGRAPHY

1. Gutman, S. G., General solution of the problems of elasticity theory in generalized cylindrical coordinates. Dokl. Akad. Nauk SSSR, Vol. 58, №6, 1947.
2. Noble, B., Application of the Wiener-Hopf Method to Solve Partial Differential Equations. Moscow, IIL, 1962.
3. Al'perin, I. G., Stresses in an infinite strip uniformly compressed along half its length. Uch. zapiski Khar'kov. Univ., Vol. 20, 1950.
4. Kogan, B. I., The state of stress of an infinite cylinder pressed into an absolutely rigid semi-infinite cylindrical sleeve. PMM Vol. 20, №2, 1956.
5. Lur'e, A. I., Three-dimensional Problems of Elasticity Theory. Moscow, Gostekhizdat, 1955.

Translated by M. D. F.

TORSION OF A TRUNCATED HYPERBOLOID

PMM Vol. 34, №2, 1970, pp. 349-354

N. A. BELOVA and Ia. S. UFLIAND
(Leningrad)

(Received November 28, 1969)

Some torsional problems are investigated which can be solved in ellipsoidal coordinates using the Mehler-Fock transformation, specially generalized for the case of an incomplete interval. The proof of the relevant inversion formula is given.

1. Formulation of the problem and its general solution. Let us consider the torsion of a two-sheeted hyperboloid of revolution, truncated at its top by an ellipsoidal surface. In degenerate ellipsoidal coordinates

$$r = c\ \mathrm{sh}\,\alpha \sin\beta, \qquad z = c\ \mathrm{ch}\,\alpha\cos\beta \tag{1.1}$$

the body which we consider occupies the region delineated by $\alpha_0 < \alpha < \infty, 0 \leqslant \beta < \beta_0$.

If the single component of an elastic displacement $v \equiv u_\varphi(\alpha,\beta)$ is taken as the basic unknown function, the problem is reduced to solving the equation [1]

$$\Delta v - r^{-2} v = 0 \tag{1.2}$$

with certain boundary conditions on the surfaces $\alpha = \alpha_0$ and $\beta = \beta_0$.

In the case when these conditions are uniform for $\alpha = \alpha_0$, two classes of problems have to be considered:

a) cross section $\alpha = \alpha_0$ is fixed, i. e.

$$v(\alpha_0, \beta) = 0$$

b) when $\alpha = \alpha_0$, shear stresses

$$\tau_{\alpha\varphi} = Gr\frac{\partial}{\partial n}\left(\frac{v}{r}\right) = 0, \qquad \text{or} \qquad \frac{\partial}{\partial\alpha}\left(\frac{v}{\text{sh}\,\alpha}\right) = 0$$

do not exist.

It is assumed that either displacement v or stress

$$\tau_{\beta\varphi} = \frac{G\sin\beta}{h}\frac{\partial}{\partial\beta}\left(\frac{v}{\sin\beta}\right), \qquad h = c\sqrt{\text{sh}^2\alpha + \sin^2\beta}$$

on the surface of the hyperboloid $\beta = \beta_0$ are prescribed.

Here G is the shear modulus and h is the Lame coefficient.

These problems can be also formulated in terms of stress function $\Phi = r^2 w$, where

$$\Delta w - 4r^{-2} w = 0 \tag{1.3}$$

In this version function Φ itself may be considered known on all those surface regions for which the stress is specified; if the displacements are specified, normal derivative of this function may be regarded as known.

We shall now consider a more general problem which consists of finding the solution for the following equation (*) :

$$\Delta u - m^2 r^{-2} u = 0, \quad 1 < x_0 < x < \infty, \quad 0 \leqslant \beta < \beta_0 \tag{1.4}$$

$$x = \text{ch}\,\alpha, \quad x_0 = \text{ch}\,\alpha_0; \qquad m = 0,1,2,\ldots$$

with the boundary conditions

$$\left(Au + B\frac{\partial u}{\partial x}\right)_{x=x_0} = 0, \quad \left(Ku + L\frac{\partial u}{\partial\beta}\right)_{\beta=\beta_0} = f(x) \tag{1.5}$$

Separation of variables in (1. 4) yields particular solutions of the following form [2]:

$$u_\nu(\alpha, \beta) = [MP_\nu^m(x) + NQ_\nu^m(x)]\,P_\nu^m(\cos\beta) \tag{1.6}$$

Applying the boundary conditions for $\alpha = \alpha_0$ and allowing for the results obtained in [3] we find $\quad u_\nu = C(\tau)\, y_\nu(x)\, P_\nu^m(\cos\beta) \qquad (\nu = -1/2 + i\tau, \quad \tau \geqslant 0)$

$$y_\nu = [AQ_\nu^m(x_0) + BQ_\nu^{m'}(x_0)]\,P_\nu^m(x) - [AP_\nu^m(x_0) + BP_\nu^{m'}(x_0)]\,Q_\nu^m(x) \tag{1.7}$$

Hence, the solution of the problem is of the following form:

$$u(\alpha, \beta) = \int_0^\infty C(\tau)\, y_\nu(x)\, P_\nu^m(\cos\beta)\, d\tau \tag{1.8}$$

Making now use of boundary condition (1. 5) for $\beta = \beta_0$ we obtain an expansion of the form

---

*) Dirichlet and Neumann problems can be reduced to analogous problems when there is no axial symmetry; similarly the boundary value problems of heat conduction in the considered region, etc.

$$f(x)=\int_0^\infty D(\tau)\,y_\nu(x)\,d\tau \tag{1.9}$$

from which we have to determine the unknown quantity

$$D(\tau)=C(\tau)\,[KP_\nu{}^m(\cos\beta_0)-L\sin\beta_0 P_\nu{}^{m'}(\cos\beta_0)] \tag{1.10}$$

Expansion (1. 9) is a generalization of the Mehler -Fock transformation for the case of an incomplete interval and of boundary conditions of the third kind (*) . In Sect. 2 of the present paper we shall prove the following inversion formula (cf. (2. 11)):

$$D(\tau)=\frac{\Gamma(^1/_2+i\tau-m)}{\Gamma(^1/_2+i\tau+m)}\frac{(-1)^m\,\tau\,\mathrm{th}\,\pi\tau}{|AQ_\nu{}^m(x_0)+BQ_\nu{}^{m'}(x_0)|^2}\int_{x_0}^\infty f(x)\,y_\nu(x)\,dx \tag{1.11}$$

which provides the final solution of our problem.

In torsional problems which can be solved by means of function v it is obviously necessary to take $m = 1$, while $B = 0$ in the case (a) and $A = \mathrm{ch}\,\alpha_0$ cs $\mathrm{ch}^2\,\alpha_0$, $B = -1$ in the case (b).

The problems in which stress function Φ is used correspond to the case when $m = 2$. Let us also note that if the boundary conditions for $\alpha = \alpha_0$ are inhomogeneous, formula (1. 11) may be applied to find the solution by the method of integral transformations (cf. , for instance, [6]).

2. Proof of the inversion formula. Theorem. Let $f(x)$ be a specified function, defined in the interval (x_0, ∞) and satisfying the following conditions:

1°. Function $f(x)$ is precise continuous and its variation is bounded in the open interval (x_0, ∞);

2°. $|f(x)|\,x^{-1/2}\ln x \in L(x_0, \infty)$

Under these conditions the following expansion is valid:

$$^1/_2[f(x-0)+f(x+0)]= \qquad (x_0>1,\ \nu=-{}^1/_2+i\tau)$$

$$=(-1)^m\int_0^\infty \frac{\tau\,\mathrm{th}\,\pi\tau\, y_\nu(x)}{|AQ_\nu{}^m(x_0)+BQ_\nu{}^{m'}(x_0)|^2}\frac{\Gamma(^1/_2+i\tau-m)}{\Gamma(^1/_2+i\tau+m)}\,d\tau\int_{x_0}^\infty f(\xi)\,y_\nu(\xi)\,d\xi \tag{2.1}$$

A and B are here real numbers of different signs.

The following estimates for $x > x_0$ are required to prove the theorem:

$$|P^m_{-1/2+i\tau}(x)|\leqslant \mathrm{ch}\,\pi\tau\,\frac{(x^2-1)^{1/2m}}{(x+1)^m}\frac{\Gamma(m+{}^1/_2)}{\Gamma(^1/_2)}P_{-1/2}(x)\leqslant$$

$$\leqslant O(1)\,\mathrm{ch}\,\pi\tau x^{-1/2}\ln x \tag{2.2}$$

$$|Q^m_{-1/2+i\tau}(x)|\leqslant \mathrm{ch}\,\pi\tau\,|Q^m_{-1/2}(x)|+{}^1/_2\,\mathrm{sh}\,\pi\tau\,|P^m_{-1/2}(x)|\leqslant O(1)\,\mathrm{ch}\,\pi\tau x^{-1/2}\ln x \tag{2.3}$$

$$|y_\nu(x)|\leqslant O(1)\,\mathrm{ch}^2\,\pi\tau x^{-1/2}\ln x \tag{2.4}$$

These estimates are obtained from the integral representations [7]

$$P^m_{-1/2+i\tau}(x)=\frac{(-1)^m}{\pi}\frac{\sqrt{2}}{\sqrt{\pi}}\,\Gamma\left(m+\frac{1}{2}\right)(x^2-1)^{m/2}\,\mathrm{ch}\,\pi\tau\int_0^\infty\frac{\cos\tau t\,dt}{(x+\mathrm{ch}\,t)^{m+1/2}}$$

---

*) Papers [3–5] dealt with similar expansion series in the case of boundary conditions of the first kind.

$$Q^m_{-1/2+i\tau}(x) = \frac{(-1)^m}{\sqrt{2\pi}}(x^2-1)^{1/2 m}\,\Gamma\left(m+\frac{1}{2}\right)\left\{\int_0^\pi \frac{\operatorname{ch}\tau t\,dt}{(x-\cos t)^{m+1/2}} - i\operatorname{sh}\pi\tau\int_0^\infty \frac{e^{-i\tau t}\,dt}{(x+\operatorname{ch}t)^{m+1/2}}\right\}$$

of the relations

$$P^{m'}_\nu(x) = \frac{1}{\sqrt{x^2-1}}P^{m+1}_\nu(x) + \frac{mx}{x^2-1}P_\nu{}^m(x)$$

$$Q^{m'}_\nu(x) = \frac{1}{\sqrt{x^2-1}}Q^{m+1}_\nu(x) + \frac{mx}{x^2-1}Q_\nu{}^m(x)$$

and estimates

$$|AQ_\nu{}^m(x_0) + BQ^{m'}_\nu(x_0)| \leqslant \operatorname{ch}\pi\tau O(1), \quad |AP_\nu{}^m(x_0) + BP^{m'}_\nu(x_0)| \leqslant \operatorname{ch}\pi\tau O(1)$$

Let us consider now the integral

$$J(T, x) = (-1)^m\int_0^T \frac{\tau\operatorname{th}\pi\tau y_\nu(x)M(\tau)}{|AQ_\nu{}^m(x_0) + BQ^{m'}_\nu(x_0)|^2}\frac{\Gamma(1/2+i\tau-m)}{\Gamma(1/2+i\tau+m)}d\tau \tag{2.5}$$

where

$$M(\tau) = \int_{x_0}^\infty f(\xi)\,y_\nu(\xi)\,d\xi \tag{2.6}$$

Integral $M(\tau)$ is a continuous function since the integrand is piecewise continuous with respect to ξ, continuous with respect to τ, and the following majorant estimate applies:

$$\int_{x_0}^\infty |f(\xi)\,y_\nu(\xi)|\,d\xi \leqslant \int_{x_1}^\infty |f(\xi)|\,O(1)\,\xi^{-1/2}\ln\xi\,d\xi < \infty \tag{2.7}$$

Using the method developed in [5] it can be shown that

$$AQ_\nu{}^m(x_0) + BQ_\nu{}^{m'}(x_0) \neq 0 \quad \text{when } \operatorname{Re}\nu \geqslant 1/2$$

We may, therefore, change the order of integration in the iterated integral (2.5) and write $J(T, x)$ as

$$J(T, x) = (-1)^m\int_{x_0}^\infty f(\xi)\,G(x, \xi, T)\,d\xi \tag{2.8}$$

$$G(x, \xi, T) = \int_0^T \frac{\tau\operatorname{th}\pi\tau y_\nu(x)\,y_\nu(\xi)}{|AQ_\nu{}^m(x_0) + BQ^{m'}_\nu(x_0)|^2}\frac{\Gamma(1/2+i\tau-m)}{\Gamma(1/2+i\tau+m)}d\tau \tag{2.9}$$

Allowing for the fact that the integrand in (2.9) is even with respect to τ, introducing a new variable $i\tau = p$ and taking into account the following relations [2]:

$$\pi\operatorname{tg}\pi p\,P^m_{p-1/2}(z) = Q^m_{-p-1/2}(z) - Q^m_{p-1/2}(z), \quad \pi\operatorname{tg}\pi p P^{m'}_{p-1/2}(z) = Q^{m'}_{-p-1/2}(z) - Q^{m'}_{p-1/2}(z)$$

we obtain

$$G(x, \xi, T) = \frac{1}{\pi i}\int_{-iT}^{iT} \frac{pQ^m_{p-1/2}(x)\,y(\xi)}{AQ^m_{p-1/2}(x_0) + BQ^{m'}_{p-1/2}(x_0)}\frac{\Gamma(1/2+p-m)}{\Gamma(1/2+p+m)}dp \quad (x \geqslant \xi) \tag{2.10}$$

$$G(x, \xi, T) = \frac{1}{\pi i}\int_{-iT}^{iT} \frac{pQ^m_{p-1/2}(\xi)\,y_{p-1/2}(x)}{AQ^m_{p-1/2}(x_0) + BQ^{m'}_{p-1/2}(x_0)}\frac{\Gamma(1/2+p-m)}{\Gamma(1/2+p+m)}dp \quad (\xi \geqslant x)$$

Since the singularities of function $\Gamma(1/2 + p - m)$ at points $p = m - 1/2, m - 3/2, .., 1/2$ are cancelled by the zeros of function $y_{p-1/2}$, the integrand in $G(x, \xi, T)$ is regular with respect to p in half-plane $\operatorname{Re}p \geqslant 0$. Hence, integration over a section of the imaginary axis can be replaced by integration (essentially, we apply here the method developed in [8]) over half-circle Γ_T where $p = Te^{i\varphi}$ and $|\varphi| \leqslant {}^1/_2\pi$.

Using the asymptotic representations given in [9] for spherical functions when $|p| \to \infty$, $|\arg p| \leqslant {}^1/_2\pi$

$$Q_{p-1/2}(\operatorname{ch}\alpha) = \left(\frac{\pi}{2p\operatorname{sh}\alpha}\right)^{1/2} e^{-p\alpha}[1 + O(|p|^{-1})] \tag{2.11}$$

$$P_{p-1/2}(\operatorname{ch}\alpha) = \left(\frac{1}{2\pi p\operatorname{sh}\alpha}\right)^{1/2}\{e^{p\alpha}[1 + \sqrt{\operatorname{ch}\alpha}\,O(|p|^{-1})] \pm ie^{-p\alpha}[1 + O(|p|^{-1})]\}$$

we can derive, after some calculations, more generalized formulas

$$Q^m_{p-1/2}(\operatorname{ch}\alpha) = (-1)^m p^{m-1/2}\left(\frac{\pi}{2\operatorname{sh}\alpha}\right)^{1/2} e^{-p\alpha}[1 + O(|p|^{-1})] \tag{2.12}$$

$$P^m_{p-1/2}(\operatorname{ch}\alpha) = \frac{p^{m-1/2}}{\sqrt{2\pi\operatorname{sh}\alpha}}\{e^{p\alpha}[1 + \sqrt{\operatorname{ch}\alpha}\,O(|p|^{-1})] \pm i(-1)^m e^{-p\alpha}[1 + O(|p|^{-1})]\} \tag{2.13}$$

$$Q^{m\prime}_{p-1/2}(\operatorname{ch}\alpha) = \frac{1}{\operatorname{sh}\alpha}(-1)^{m+1}p^{m+1/2}\left(\frac{\pi}{2\operatorname{sh}\alpha}\right)^{1/2} e^{-p\alpha}[1 + O(|p|^{-1})] \tag{2.14}$$

$$P^{m\prime}_{p-1/2}(\operatorname{ch}\alpha) = \frac{p^{m+1/2}}{\operatorname{sh}\alpha\sqrt{2\pi\operatorname{sh}\alpha}}\{e^{p\alpha}[1 + \sqrt{\operatorname{ch}\alpha}\,O(|p|)^{-1})] \pm i(-1)^{m+1} e^{-p\alpha}[1 + O(|p|^{-1})]\} \tag{2.15}$$

Then for $x \geqslant \xi$

$$G(x, \xi, T) = \frac{1}{2\pi i}\frac{(-1)^m}{\sqrt{\operatorname{sh}\alpha\operatorname{sh}\gamma}}\int_{\Gamma_T}\{e^{-p(\alpha-\gamma)} - (-1)^k e^{-p(\alpha+\gamma-\alpha_0)} +$$

$$+ e^{-p(\alpha-\gamma)}\sqrt{\operatorname{ch}\gamma}\,O(|p|^{-1}) + e^{-p(\alpha+\gamma-2\alpha_0)}O(|p|^{-1}) + e^{-p(\alpha+\gamma)}O(|p|^{-1})\}\,dp =$$

$$= \frac{1}{\pi}\frac{(-1)^m}{\sqrt{\operatorname{sh}\alpha\operatorname{sh}\gamma}}\left\{\frac{\sin T(\alpha-\gamma)}{\alpha-\gamma} - (-1)^k\frac{\sin T(\alpha+\gamma-2\alpha_0)}{\alpha+\gamma-2\alpha_0} + \right.$$

$$\left. + \sqrt{\operatorname{ch}\gamma}\,O(1)J_1 + O(1)J_2 + O(1)J_3\right\} \tag{2.16}$$

where

$$x = \operatorname{ch}\alpha,\quad \xi = \operatorname{ch}\gamma,\quad x_0 = \operatorname{ch}\alpha_0,\quad k = \begin{cases}0 \text{ when } B = 0\\ 1 \text{ when } B \neq 0\end{cases}$$

$$|J_i| \leqslant \int_0^{\pi/2} e^{-T\lambda_i\cos\varphi}\,d\varphi \leqslant \frac{\pi}{2}\frac{1 - e^{-\lambda_i T}}{\lambda_i T} \tag{2.17}$$

$$\lambda_1 = \alpha - \gamma,\ \lambda_2 = \alpha + \gamma - 2\alpha_0,\ \lambda_3 = \alpha + \gamma$$

In the case of $\xi \geqslant x$, i.e. $\gamma \geqslant \alpha$, it is necessary to change from α to γ and from γ to α in (2.16). Hence, (2.8) is then written out as

$$J(T, x) = (-1)^m\int_{\alpha_0}^{\alpha} f(\operatorname{ch}\gamma)\operatorname{sh}\gamma\,G(x, \xi, T)\,d\gamma +$$

$$+ (-1)^m\int_{\alpha}^{\infty} f(\operatorname{ch}\gamma)\operatorname{sh}\gamma\,G(x, \xi, T)\,d\gamma = I_1 + I_2 \tag{2.18}$$

We split the integration interval in integral I_1 into intervals $(\alpha_0, \alpha - \delta)$ and $(\alpha - \delta, \alpha)$ and choose first a sufficiently small δ and then a sufficiently great T. From the Dirichlet theorem, allowing for conditions 1° and 2°, we have for $T \to \infty$ the following expressions:

$$\int_{\alpha_0}^{\alpha-\delta} f(\operatorname{ch}\gamma)\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{\sin T(\alpha-\gamma)}{\alpha-\gamma}\,d\gamma = o(1)$$

$$\frac{1}{\pi}\int\limits_{\alpha-\delta}^{\alpha} f(\operatorname{ch}\gamma)\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{\sin T(\alpha-\gamma)}{\alpha-\gamma}\,d\gamma = {}^1/_2 f(\operatorname{ch}\alpha-0)+o(1)$$

$$\int\limits_{\alpha_0}^{\alpha} f(\operatorname{ch}\gamma)\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{\sin T(\alpha+\gamma-2\alpha_0)}{\alpha+\gamma-2\alpha_0}\,d\gamma = o(1)$$

$$\int\limits_{\alpha-\delta}^{\alpha} |f(\operatorname{ch}\gamma)|\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{1-e^{-T(\alpha-\gamma)}}{T(\alpha-\delta)}\,d\gamma = o(1)$$

$$\int\limits_{\alpha_0}^{\alpha-\delta} |f(\operatorname{ch}\gamma)|\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{1-e^{-T(\alpha-\gamma)}}{T(\alpha-\gamma)}\,d\gamma = O(T^{-1})$$

$$\int\limits_{\alpha_0}^{\alpha} |f(\operatorname{ch}\gamma)|\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{1-e^{-T(\alpha+\gamma-2\alpha_0)}}{T(\alpha+\gamma-2\alpha_0)}\,d\gamma = O(T^{-1})$$

$$\int\limits_{\alpha_0}^{\alpha} |f(\operatorname{ch}\gamma)|\left(\frac{\operatorname{sh}\gamma}{\operatorname{sh}\alpha}\right)^{1/2}\frac{1-e^{-T(\alpha+\gamma)}}{T(\alpha+\gamma)}\,d\gamma = O(T^{-1})$$

Thus, when $T\to\infty$ $I_1\to {}^1/_2 f(x-0)$. In a similar manner it can be shown that $\lim I_2 = {}^1/_2\, f(x+0)$ when $T\to\infty$; this proves expansion (2.1).

3. Example. Let us consider a truncated hyperboloid adhering in the area $\alpha=\alpha_0$ to an immobile rigid stamp and subjected to torsion by shear stresses $\tau_{\beta\varphi}=F(\alpha)$ applied to the surface $\beta=\beta_0$. Boundary conditions for the displacement $v(\alpha,\beta)$ have in this case the following form:

$$v\Big|_{\alpha=\alpha_0}=0,\quad \frac{\partial}{\partial\beta}\frac{v}{\sin\beta}\Big|_{\beta=\beta_0}=f(\alpha)=\frac{c\sqrt{\operatorname{sh}^2\alpha+\sin^2\beta_0}}{G\sin\beta_0}F(\alpha) \qquad (3.1)$$

The solution of this problem is given by the formula

$$v=\int\limits_0^\infty C(\tau)\,P_\nu^{\,1}(\cos\beta)\,y_\nu(x)\,d\tau \qquad (3.2)$$

where on the basis of (1.7), (1.10) and (1.11)

$$y_\nu(x)=Q_\nu^{\,1}(x_0)\,P_\nu^{\,1}(x)-P_\nu^{\,1}(x_0)\,Q_\nu^{\,1}(x),\quad \nu={}^1/_2+i\pi \qquad (3.3)$$

$$C(\tau)=\frac{\tau\sin\beta_0\operatorname{th}\pi\tau}{(\tau^2+{}^1/_4)\,[\sin^2\beta_0 P_\nu^{\,1\prime}(\cos\beta_0)+\cos\beta_0\,P_\nu^{\,1}(\cos\beta_0)]}\int\limits_{\alpha_0}^{\infty} f(\alpha)\,y_\nu(x)\operatorname{sh}\alpha\,d\alpha \qquad (3.4)$$

In particular, when a linear load of intensity q is applied to the circumference $\alpha=\alpha^*$, the integral in (3.4) can be calculated and its value is

$$\frac{q\operatorname{sh}\alpha^*}{G\sin\beta_0}y_\nu(\operatorname{ch}\alpha^*)$$

The final solution is given by a single quadrature (3.2).

BIBLIOGRAPHY

1. Arutiunian, N. Kh. and Abramian, B. L., Torsion of Elastic Bodies. Fizmatgiz, 1963.
2. Lebedev, N. N., Special Functions and Their Applications. 2nd Edition. Fizmatgiz, 1963.
3. Belova, N. A. and Ufliand, Ia. S., Dirichlet problem for a toroidal segment. PMM Vol. 31, №1, 1967.
4. Ulitko, A. F., A generalization of the Mehler-Fock integral transformation. Prikl. mekh., Vol. 3, №5, 1967.
5. Belova, N. A. and Ufliand, Ia. S., Expansion in eigenfunctions of a certain singular boundary value problem of the Legendre equation. Differentsial'nye uravneniia, Vol. 3, №8, 1967.
6. Grinberg, G. A., Selected Problems of the Mathematical Theory of Electrical and Magnetic Phenomena. M.-L., Izd. Akad. Nauk SSSR, 1948.
7. Bateman, G. and Erdely, A., Higher Transcendental Functions Vol. 1 (Russian translation), M., "Nauka", 1965.
8. Lebedev, N. N. and Skal'skaia, I. P., Integral expansion of an arbitrary function in terms of spherical functions. PMM Vol. 30, №2, 1966.
9. Belova, N. A., An integral expansion in terms of spherical functions of the first and second kind. Differentsial'nye uravneniia, Vol. 5, №11, 1969.

Translated by J. M. S.

CONTACT PROBLEM FOR AN ELASTIC HALF-PLANE AND A SEMI-INFINITE ELASTIC ROD ADHERING TO IT

PMM Vol. 34, №2, 1970, pp. 354-359
V. L. VOROB'EV and G. Ia. POPOV
(Odessa)
(Received June 26, 1969)

The problem which will be considered is as follows. An elastic semi-infinite rod having constant cross section F is glued (or welded) to the side of an elastic semi-infinite plate having thickness h (see Fig. 1). At an arbitrary distance b from the end-face of the rod a unit force is applied in the direction of the rod axis. The contact shear stress $\tau_0(x)$ and the normal stress $\sigma_0(x)$ in an arbitrary cross section of the rod are to be found, assuming that the rod is not subjected to any bending moments (normal contact stress is not taken into account). A similar problem for an infinite rod was solved in [1]. The case of a semi-infinite rod was considered in [2, 3]; in [2] an approximate solution was given, while in [3] an exact solution was obtained

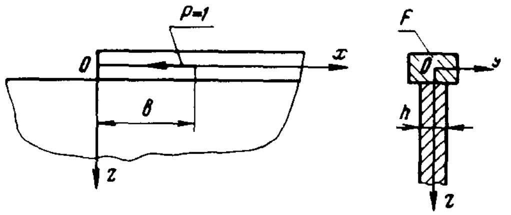

Fig. 1